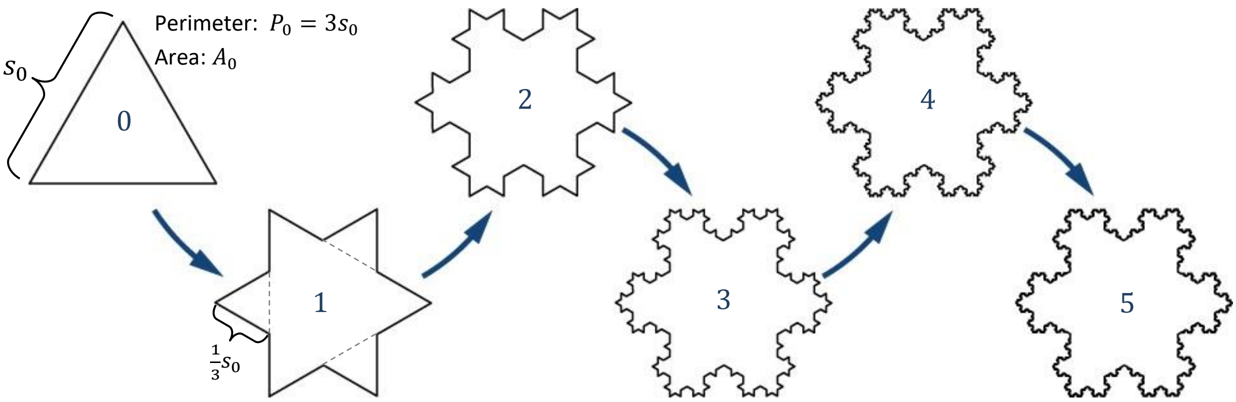
Fractal geometry

The shape of the bifurcation diagram has features like the Cantor set and self-similarity, properties that are typical of forms that arise in non-linear systems. This section will study such forms, that come together under the general term: fractal geometry. A proper definition will be given once the required concepts are explained.

Self-similarity

A shape is **self-similar** if it contains parts that look like a miniature copy of the whole shape, which means that these parts in turn must contain smaller copies of themselves ad infinitum. A good illustration of self-similarity and fractal geometry is the **Koch snowflake**.



Step 0 Start with an equilateral triangle with side $s_0 = 1$, perimeter $P_0 = 3s_0$ and area A_0.

Step 1 Put an equilateral triangle of side length $s_0/3$ with the base at the mid section of each side, pointing outward and remove the base.

Each side of figure 0 has increased by a factor $\frac{4}{3} \to P_1 = \frac{4}{3}P_0$ and the area is $A_1 = A_0 + 3\frac{A_0}{9}$.

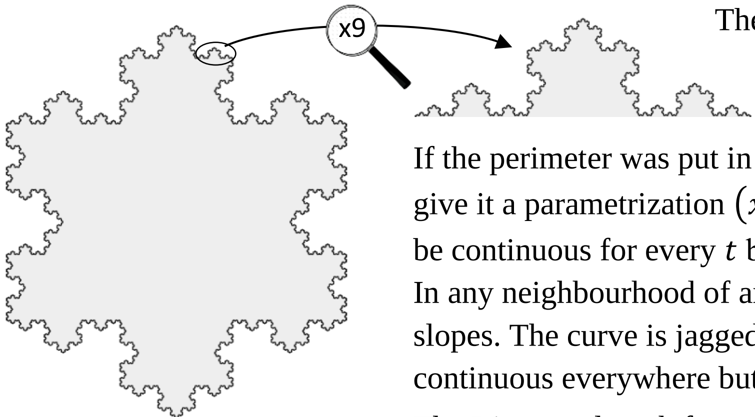
Step 2 Repeat the process for each of the $3 \cdot 4 = 12$ segments, 4 times as many with unit a third in size.

Each segment is increased in length by a factor $4/3 \to P_2 = (4/3)^2 P_0$.

Step n Repeat the process and the perimeter will be $P_n = P_0 \cdot (4/3)^n$ and the area becomes:

$$A_n = A_0\left(1 + 3 \cdot \frac{1}{9} + 4 \cdot 3 \cdot \frac{1}{9^2} + \cdots + 4^{n-1} \cdot 3 \cdot \frac{1}{9^n}\right) = \frac{A_0}{3}\left(3 + (1 + k + k^2 + \cdots + k^{n-1})\right) = \frac{A_0}{3}\left(3 + \frac{1-k^n}{1-k}\right) \text{ with } k = \frac{4}{9}$$

In the limit $n \to \infty$ there is a shape called the Koch snowflake with a continuous perimeter of infinite length, $P_n = (4/3)^n \cdot P_0 \to \infty$ as $n \to \infty$. The area inside the perimeter is however finite $A_\infty = \frac{8}{5}A_0$.



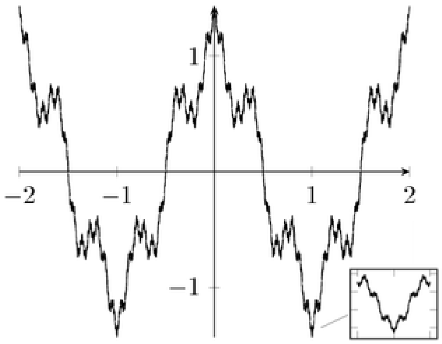
Take a small segment and magnify it by a factor 3^n. The curvy line will appear elsewhere in the figure as long as the opposite ends are no further apart than one unit. The figure is self-similar.

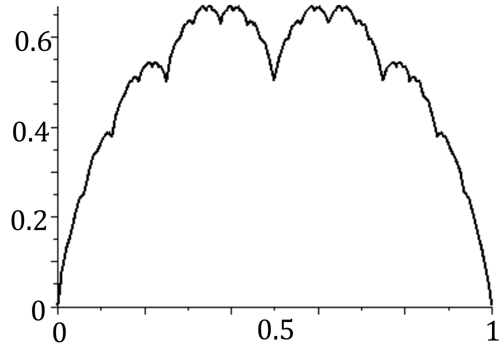
If the perimeter was put in a coordinate system it would be no problem to give it a parametrization $\big(x(t), y(t)\big)$ for $t \in [0,1]$. $x(t)$ and $y(t)$ would be continuous for every t but it wouldn't have a tangent in a single point. In any neighbourhood of any point there are secants with very different slopes. The curve is jagged in every magnification. $x(t)$ and $y(t)$ are continuous everywhere but differentiable nowhere.

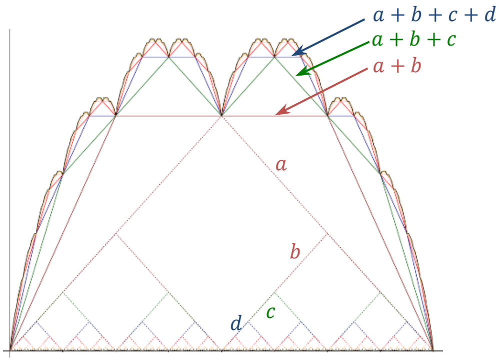
The Riemann length formula $\Delta L = \int \sqrt{\dot{x}^2 + \dot{y}^2}dx$ doesn't work.

The Koch snowflake perimeter belongs to a class of objects that are called **pathological** in contrast to more traditional objects that are called **well-behaved**.

Other pathological objects are the Weierstrass and the Blancmange function. They are both self-similar, continuous everywhere and differentiable nowhere. Pathological functions are more common than well-behaved functions but less useful. Well-behaved + pathological function gives a new pathological function.







$$f(x)=\sum_{k=0}^{\infty} a^k \cos\left(b^k \pi x\right) \qquad \begin{array}{l} a \in]0,1[\\ b \in 2\mathbb{Z}^+ + 1 \\ ab > 1 + 3\pi/2 \end{array}$$

$$f(x)=\sum_{k=0}^{\infty} \frac{\min_{n\in\mathbb{Z}} \left|2^k x - n\right|}{2^n}$$

Curves that have detail at every zoom-level will never approach a straight line when zoomed in on.
They are called **fractal curves** and geometrical shape with such properties are a part of **fractal geometry**.

Pathological objects like these were studied in the 1800's. They caused a crisis in mathematics, the old geometric intuition could no longer be used, it could and sometimes did lead to false assumptions.
A new and more secure foundation weas needed with stricter definitions and stronger criteria for valid proofs.

Foundations

A **metric space** $M = (U, d)$ consists of a set U of points with
a metric d that gives a distance $d(x, y)$ between all points of U that satisfies:

1. $d(x, y) = 0$ — Identity of points at distance 0
2. $d(x, y) = d(y, x)$ — Symmetry
3. $d(x, y) \leq d(x, z) + d(z, y)$ — Triangle inequality

$d(x, y) \geq 0$ follows from 1. 2. and 3.

Terms**:**
Open ball with centre p and radius $r > 0$ is the set $B_r(p) = \{x \in S | d(x, p) < r\}$
Neighbourhood V_x of point x is any set where there is an open ball $B_r(x)$ contained in V_x.
Isolated point x of a subset $S \subseteq U$ is a point in S with a neighbourhood V_x with no other points in S than x.
Discrete set, a set made up of isolated points.

$$\left.\begin{array}{l} \textbf{Limit point} \\ \textbf{Cluster point} \\ \textbf{Accumulation point} \end{array}\right\} \begin{array}{l} \text{of a set } S \text{ is a point } x \text{ where every neighbourhood } V_x \\ \text{contains a point of } S \text{ other than } x \text{ itself.} \end{array}$$

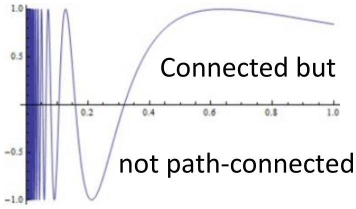
Open set S is a set where each point $p \in S$ is the centre of an open ball contained in S.
Closed set S is a set whose complement is open, or equivalently a set that contains all its limit points.
Interior point x of a set S is a point that has an open ball centered at x that is contained in S.
Interior of a set S is denoted $S°$. It is the set of points that are interior to S.
Closure of a set S is denoted $\overline{S}$, it contains S and all its limit points.
Boundary of a set S is denoted ∂S. It is the closure of S minus the boundary, $\partial S = \overline{S} \setminus S°$.

Connected set can't be represented as the union of two or more disjoint non-empty open subsets.
Totally disconnected means a set whose only connected subsets are one-point sets.
Path-connected, if every pair of points p_1, p_2 can be connected by a continuous path:

$$\exists f : f \in C([0,1]),\, f(0) = p_1,\, f(1) = p_2$$

An example of a connected but not path-connected set is:

$$\{(x, \sin x^{-1}) : x \in \mathbb{R}\} \cup \{(0, y) : y \in [0,1]\}$$

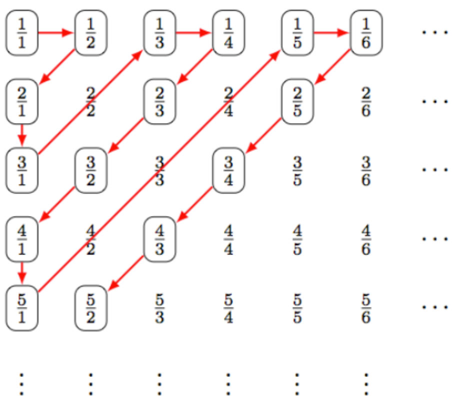


Dense, a set S in U is dense if $\overline{S} = U$. The set and its limit points fill the metric space, $\mathbb{Q}$ is dense in $\mathbb{R}$.
Nowhere dense, a set whose closure has empty interior. $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$.
Meagre, a set is meagre if it's the countable union of nowhere dense subsets of U.
Cardinality, is a measure of the number of elements in finite and infinite sets.

Two sets have the same cardinality if there is a one-to-one pairing (bijective function) from elements of one set to elements of the other set. The integers and the rational numbers have the same cardinality, $|\mathbb{Z}| = |\mathbb{Q}|$ since the rationals can be put on a line and be counted q_1, q_2, q_3. Sets with this cardinality are **countable**, $|\mathbb{Z}| = \aleph_0$.

The cardinality of the real numbers is higher than that of the integers, they can't be labelled $r_1, r_2, \ldots$. Sets with higher cardinality than $\aleph_0$ are **uncountable**. If there was a way to put the real numbers in a sequence you could use Cantor's diagonal argument to construct a real number that was not in the list. $|\mathbb{R}| = c > \aleph_0$ where c is the cardinality of the continuum.

$\mathbb{R}$ and $\mathbb{R}^n$ have the same cardinality, $|\mathbb{R}| = |\mathbb{R}^n|$. A bijective function can be made by interweaving the digits of a base-n representation of the reals.



s_1	$= 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \ldots$
s_2	$= 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1 \ldots$
s_3	$= 0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0 \ldots$
s_4	$= 1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1 \ldots$
s_5	$= 1\,1\,0\,1\,0\,1\,1\,0\,1\,0\,1 \ldots$
s_6	$= 0\,0\,1\,1\,0\,1\,1\,0\,1\,1\,0 \ldots$
s_7	$= 1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0 \ldots$
s_8	$= 0\,0\,1\,1\,0\,0\,1\,1\,0\,0\,1 \ldots$
s_9	$= 1\,1\,0\,0\,1\,1\,0\,0\,1\,1\,0 \ldots$
s_{10}	$= 1\,1\,0\,1\,1\,1\,0\,0\,1\,0\,1 \ldots$
s_{11}	$= 1\,1\,0\,1\,0\,1\,0\,0\,1\,0\,0 \ldots$
$\vdots$	$\vdots \; \vdots \; \vdots \; \vdots \; \vdots \; \vdots \; \vdots \; \vdots \; \vdots \; \vdots \; \vdots \; \ddots$

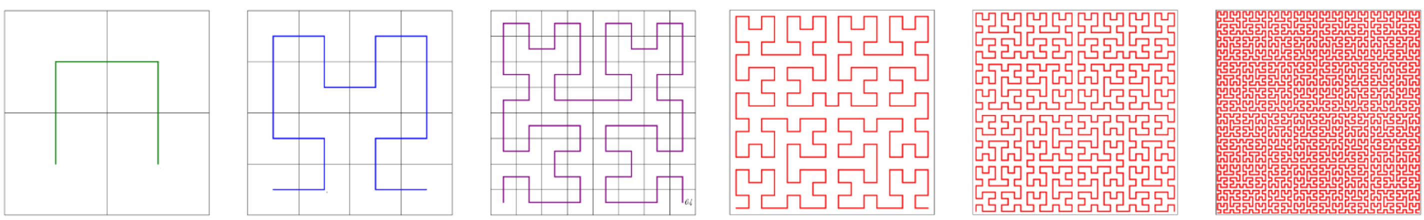
s	$= 1\,0\,1\,1\,1\,0\,1\,0\,0\,1\,1 \ldots$

$$\begin{matrix} \mathbb{R}^2 & & \mathbb{R} \\ \begin{pmatrix} s_1 s_2 s_3 \ldots \\ t_1 t_2 t_3 \ldots \end{pmatrix} & \rightleftarrows & s_1 t_1 s_2 t_2 \ldots \end{matrix}$$

The set of all subsets of S is called the **power set**, $P(S)$. A subset is described by yes/no questions of membership for the elements of S. Cantor's diagonal argument shows that $P(S)$ is bigger than S, $|P(S)| = 2^{|S|} > |S|$.

$|\mathbb{R}| = |\mathbb{R}^2|$ raises the question: Is there a continuous, bijective function from $[0,1]$ to $[0,1] \times [0,1]$?
This is not possible, but if you lower the demands a little and allow the curve to visit a point more than once then it's possible to have a curve that completely covers a square, a **space-filling curve**.

The first such example was constructed by Peano. Another space-filling curve with a similar construction is the **Hilbert curve**. In the limit as $n \to \infty$ is a curve that covers the unit square.



$n = 1$ 2 3 4 5 6

The cardinality of the irrationals $|\mathbb{R}\backslash\mathbb{Q}| = |\mathbb{R}| > |\mathbb{Q}|$ means that the fraction of rational numbers among the real numbers is zero and a zillion times more of them wouldn't increase their number.

Would it not be reasonable to have: $f_{\mathbb{R}\backslash\mathbb{Q}}(x) = \begin{cases} 1\ if\ x \in \mathbb{R}\backslash\mathbb{Q} \\ 0\ if\ x \in \mathbb{Q} \end{cases} \quad \rightarrow \quad \int_0^1 f_{\mathbb{R}\backslash\mathbb{Q}}(x)\,dx = 1$

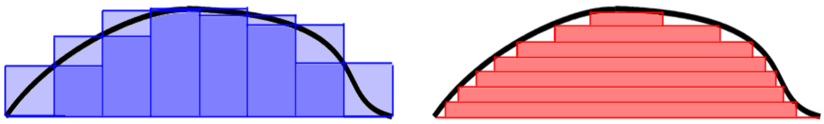
The Riemann definition of an integration is based on approximating an area with inscribed and circumscribed polygons. Technically it can be done with stepwise constant functions, $\overline{\varphi}$ above f and $\underline{\varphi}$ below f. The integral of f over $[0,1]$ is defined if the upper and lower integrals converge to the same limit:

$$\underline{I} \equiv \sup_{\underline{\varphi}} \int_0^1 \underline{\varphi}(x)\,dx \quad \overline{I} \equiv \inf_{\overline{\varphi}} \int_0^1 \overline{\varphi}(x)\,dx \qquad If\ \underline{I} = \overline{I} \text{ then } \int_0^1 f(x)dx \text{ is defined as that value}$$

$$\begin{matrix} \overline{I}(f_{\mathbb{R}\backslash\mathbb{Q}}) = 1 \\ \underline{I}(f_{\mathbb{R}\backslash\mathbb{Q}}) = 0 \end{matrix} \Rightarrow \int_0^1 f_{\mathbb{R}\backslash\mathbb{Q}}(x)dx \text{ is not defined in the Riemann version of integration.}$$

There is another definition called **Lebesgue integration** that can integrate a larger class of functions. The definition focuses on partitioning the y-axis instead of the x-axis.

With Lebesgue integration: $\int_0^1 f_{\mathbb{R}\backslash\mathbb{Q}}(x)dx = 1$



The **Lebesgue measure** of a set $\mu(S)$ is given by $\int_{\mathbb{R}} f_S(x)dx$ with indicator function $f_S(x) = \begin{cases} 1\ if\ x \in S \\ 0\ if\ x \backslash\notin S \end{cases}$

$\mu([a,b]) = b - a$ With or without endpoints in the interval.

$\mu\left(\bigcup_{k\in K} I_k\right) = \sum_{k\in K} \mu(I_k)$ With adjoint intervals I_k over a finite or countable index set K.

The Lebesgue can't give a length to every subset of $\mathbb{R}$ or an area to every subset of $\mathbb{R}^2$.
In 1905 it was shown by Vitalis that you can construct sets by using the axiom of choice that **non-measurable**.
A Vitali set V is hard to visualize. It's uncountable and every difference of different elements is irrational.
All reasonable subsets of $\mathbb{R}$ have a Lebesgue measure, they are **measurable**.

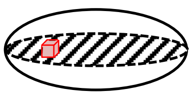
A set of Lebesgue measure zero is called a **null set**, it can be covered by a countable union of intervals of arbitrarily small total length. A property that holds for all elements of a set except for a null set is said to hold **almost everywhere** (a.e.).

Dimension

What is the dimension of a curve that passes through every point of a square; is it one, two or something in between? One way to get at the dimension of an object is to see how many more units (of length/area/volume) it takes to cover if the scale of the unit is decreased.

A line: Divide the yardstick into n pieces and it takes n times more units to cover the line.
If you cover it with squares or cubes and reduce every side a factor n^{-1}, it still takes n times more units.

A rectangle:  Divides the sides of the unit by n and it takes n^2 as many units to cover it.

A volume:  Reduce the scale of the unit by a factor $1/n$ and it takes n^3 as many units to cover.

Divide the unit-scale by a factor n and you need n^D units to cover object.

Dimension: $D = \dfrac{\log n^D}{\log n} = -\dfrac{\log N}{\log \varepsilon}$ N: Increase factor for boxcount
ε: Scale factor for unit

Ex: The perimeter of the Koch snowflake.
Divide the measuring rod in 3 pieces and you can measure more details that take 4 times as many units to cover.
Dimension $D = \log 4 \,/ \log 3 = 1.26 \ldots$
The dimension derived by similarity for self-similar object is usually not an integer.

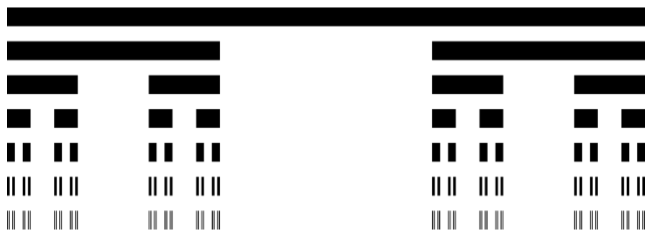
Ex: Dimension of the Hilbert curve.
Step 1: $2^2 \cdot 2 - 2$ segments of length $1/4$ (Each square contains two line segments except the first and last)
Step 2: $4^2 \cdot 2 - 2$ segments of length $1/8$
$\ldots$
Step n: $(2^n)^2 \cdot 2 - 2$ segments of length $1/2^{n+1}$

$\varepsilon = 1/2$
$N \to 2^{2(n+1)}/2^{2n} = 2^2$ $D \to -\dfrac{\log 2^2}{\log 1/2} = 2$
The Hilbert curve has dimension 2.

Cover a set $S \subset \mathbb{R}^n$ with n-dimensional cubes with side length δ. Let N_δ be the smallest number of cubes to make such a covering, then the **box dimension** of S is defined as: $D_B(S) = \lim\limits_{\delta \to 0} \dfrac{\log N_\delta(S)}{-\log \delta}$

The dimension of a point is 0 and the dimension of $\mathbb{Q} \cap [0,1]$ is 1.
What kind of object might have a dimension in between?

The **Cantor set** is a common object in mathematics.
If the real numbers in $[0,1]$ is expressed in base-3 then the set consists of those points $0.\sigma_1\sigma_2\sigma_3 \ldots$ with $\sigma_k \in \{0,2\}$.
Remove the middle third and repeat for each line segment, do it repeatedly and takt the limit to get the Cantor set C.

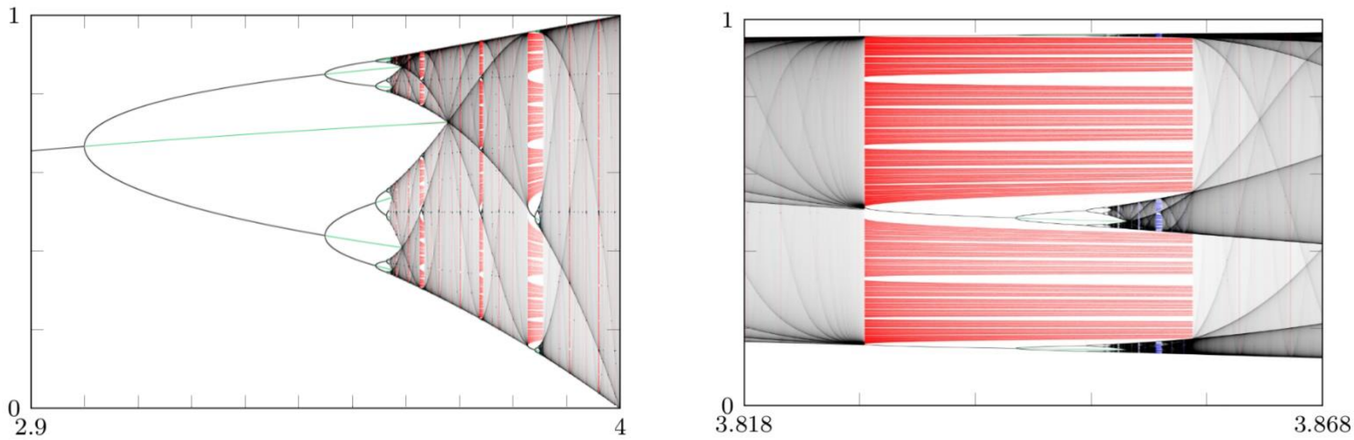


The similarity dimension of the Cantor set becomes $D = -\log 2 \,/ \log 3^{-1} = \log 2 \,/ \log 3 = 0.6309 \ldots$
It's totally disconnected, the Lebesgue measure is zero and the cardinality is that of the continuum.
Replace all digits equal to 2 with 1, shift from ternary to binary base and $[0,1]$ is retrieved, one-to-one.

In a variation of the Cantor set you remove less by a factor k in each step. The total removed length is:

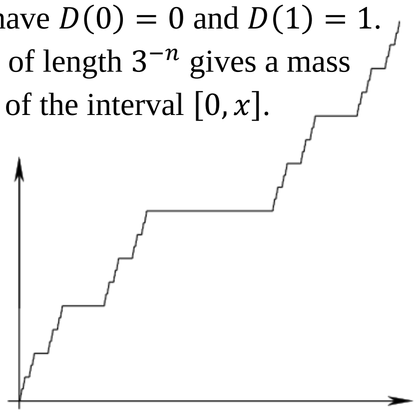
$$3^{-1} + 2 \cdot 3^{-2} \cdot k + 2^2 \cdot 3^{-3} \cdot k^2 + \cdots = \frac{1}{3} \cdot \left(1 + \frac{2k}{3} + \left(\frac{2k}{3}\right)^2 + \cdots\right) = \frac{1}{3} \cdot \frac{1}{1 - \frac{2k}{3}} = \frac{1}{3 - 2k}$$

With $k = 0.5$ you only remove half and the remaining part has Lebesgue measure 0.5. This is a **fat Cantor set**.



Bifurcation diagram with attracting sets in shades of gray depending on density of attractor, repelling periodic orbits in green and repelling Cantor sets in red. Nested repelling Cantor set in blue in period-3 window.
The Hausdorff dimension of the Feigenbaum attractor at r_{2^∞} when chaos first appear is $D_H = 0.538 \ldots$.

The Cantor set is used in the construction of another 'pathological' function, called the **devil's staircase** $D(x)$, continuous, increasing with $D'(x) = 0$ almost everywhere and still manages to have $D(0) = 0$ and $D(1) = 1$. Distribute a unit mass homogeneously over step n in the Cantor set. 2^n intervals of length 3^{-n} gives a mass density of $3^n 2^{-n}$ in each segment. Integrate from 0 to x to get $D(x) =$ the mass of the interval $[0, x]$.



$$\rho_n(x) = \begin{cases} (3/2)^n \; if \; x \in C_n \\ 0 \; if \; x \in C_n \end{cases} \qquad D_n(x) \equiv \int_0^x \rho_n(x')dx'$$

$$D(x) \equiv \lim_{n\to\infty} D_n(x) \quad \text{Devil's staircase or the Cantor function}$$

The derivative $D'(x)$ is discontinuous at an uncountable number of places. Devil's staircase has many applications in physics.

The box dimension D_B is not the only way and not the best way to define the dimension of complex shapes. There is an alternative called the **Hausdorff dimension** D_H, more complex to define and harder to calculate but with more reasonable dimension when $D_B \neq D_H$. Another name is the Hausdorff-Besicovitch dimension.

Cover a set S with small n-dimensional volumes v_i and let d_i be the maximal distance between points in v_i and let $\delta = \max_i d_i$. For each δ and for each $s \in \mathbb{R}^+$ choose the volumes V_i so that $\sum_i d_i^s$ is minimised.

The limit $H_s = \lim_{\delta\to 0} \sum_i d_i^s$ depends on the value of s. There is a critical value D_H.

$$s < D_H \Rightarrow H_s = \infty$$
$$s > D_H \Rightarrow H_s = 0$$

D_H is the Hausdorff dimension.

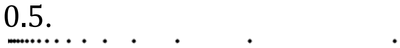
$Ex.$ Divide a line segment of length L into N pieces each with length $\delta = L/N \to N = L/\delta$.

$$H_S = \lim_{\delta\to 0} \sum_i d_i^s = \lim_{\delta\to 0}\left(\frac{L}{\delta}\cdot\delta^s\right) = L \cdot \lim_{\delta\to 0}\delta^{s-1} \quad \to \quad \begin{cases} H_s = \infty \; if \; s < 1 \\ H_s = 0 \; if \; s > 1 \\ H_s = L \; if \; s = 1 \end{cases} \quad \to \quad D_H = 1$$

D_H is calculated with cover sets that can have any shape whereas D_B uses identical sized squares, cubes etc. which can lead to the 'wrong' dimension.

The limit of this sequence is a shape where $D_H \neq D_B$

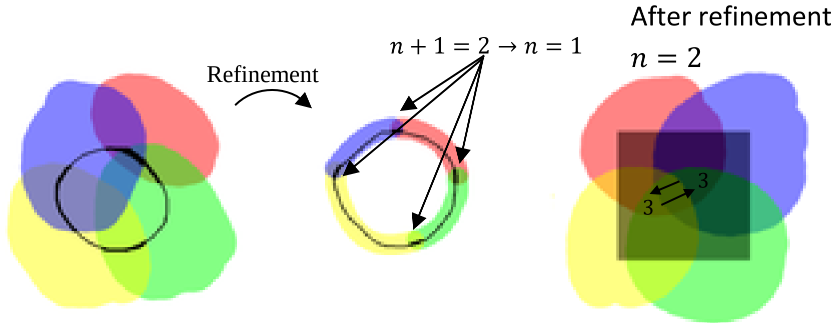
Another example with $D_H \neq D_B$ is $S = \{1/n | n \in \mathbb{Z}^+\}$ where $D_H = 0$ and $D_B = 0.5$.



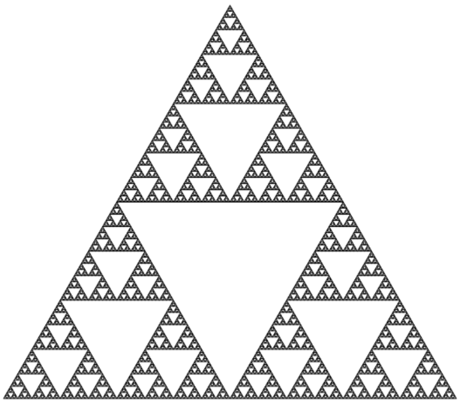
Dimension zero is a more reasonable dimension than 0.5 for a discrete set of points.

Fractal geometry is the study of geometric objects with details at every scale magnification. This is expressed by a Hausdorff dimension that exceeds the object's natural dimension. The Koch snowflake perimeter can be deformed continuously into a circle so in a topological sense the dimension should be 1. This is the topological dimension.

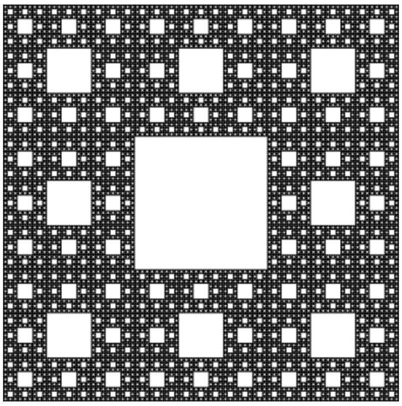
The **topological dimension,** also called the **Lebesgue covering dimension** is defined by covering the object with a union of open sets. The covering dimension is the smallest n for which every cover has a refinement such that no point lies in more than $n + 1$ covering sets.



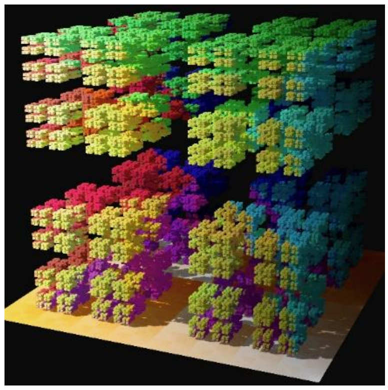
Fractals in mathematics



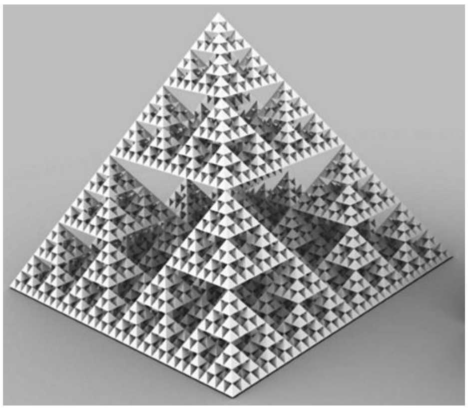
Sierpinski triangle, $D_H = 1.585$



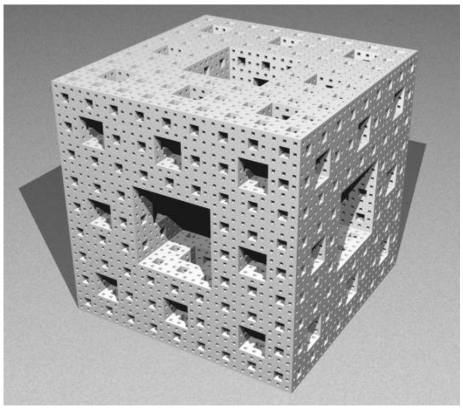
Sierpinski carpet, $D_H = 1.893$



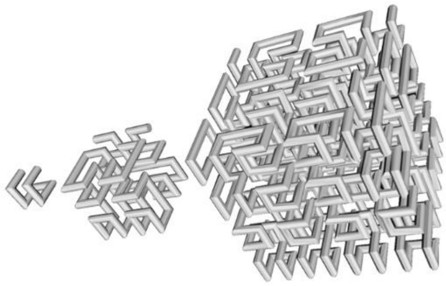
Cantor dust, $D_H = 1.893$



Fractal pyramid, $D_H = 2.322$



Menger sponge, $D_H = 2.727$



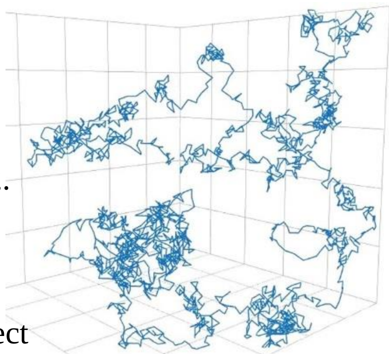
3D Hilbert curve, $D_H = 3$

Fractals in nature

In nature there is always an upper limit for magnifications but self similarity can still apply in a limited interval of scale, as can be seen in a fern or a broccoli.

Many mathematical models lead to shapes and patterns that have fractal properties. An example is Brownian motion and diffusion that can be described by **random walks**.

Let a particle travel a certain distance between collisions where direction is changed at random. The expected distance after n steps grows as $\sqrt{n}$, the same for all dimensions. With step size δ you need L/δ^2 steps for distance L, the dimension is 2.



Hofstaedter's butterfly describes energy levels of electrons on a 2D lattice in a magnetic field. It plays an important role in the quantum Hall effect and the description of topological quantum numbers.